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Wilson loops in $\mathcal{N} = 6$ superspace for ABJM theory

M. Rosso,^a C. Vergu^a

^a*ETH Zürich, Institut für Theoretische Physik,
Wolfgang Pauli Strasse 27,
CH-8093 Zürich, Switzerland*

E-mail: mrosso@itp.phys.ethz.ch, verguc@itp.phys.ethz.ch

ABSTRACT: In this paper we construct a light-like polygonal Wilson loop in $\mathcal{N} = 6$ superspace for ABJM theory. We then use it to obtain constraints on its two- and three-loop bosonic version, by focusing on higher order terms in the θ expansion. The Grassmann expansion of the three-loop answer contains integrals which may be elliptic polylogarithms. Our results take their simplest form when expressed in terms of $OSp(6|4)$ supertwistors.

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1 Introduction

The fact that a certain operator or observable can be supersymmetrized often has important consequences. Even for an observable which is not supersymmetric but admits a supersymmetric completion it can be useful to consider it. An early example of the usefulness of supersymmetry for a non-supersymmetric quantity is the argument of ref. [1] which uses supersymmetry to show that some tree-level helicity amplitudes in Yang-Mills theory vanish. Then Parke and Taylor used supersymmetry to simplify the computations of some non-vanishing helicity amplitudes (see refs. [2–4]).

The observables we will consider in this paper are light-like polygonal Wilson loops in ABJM theory [5] (see also refs. [6–12] for earlier related work). The ABJM theory is a theory of two Chern-Simons gauge connections interacting with fermions and scalars in bifundamental representations. It is a superconformal theory with symmetry group $OSp(6|4)$. It is therefore natural to build observables which are invariant under this superconformal symmetry.

The shape of a general Wilson loop changes in a complicated way under conformal transformations. The shape of a polygonal light-like contour transforms in a much simpler way which makes it more attractive to study (see however ref. [13], where a general Wilson loop shape is considered).

In pure Chern-Simons theory, appropriately regularized (framed) Wilson loops yield topological invariants. When adding matter fields, like in the ABJM theory, the Wilson loops start depending on the shape of their contour in a more complicated

way. In Yang-Mills theory, the Wilson loop require a non-trivial renormalization as explained in refs. [14, 15] for the Euclidean signature and later in ref. [16] for Lorentzian signature. If the contour of a Wilson loop in Chern-Simons theory has continuous third derivatives, the one-loop perturbative correction is finite (see for example ref. [17]) but not topological invariant.

When introducing singularities, and just like their Yang-Mills counterpart, the ABJM light-like Wilson loops become UV-divergent and these UV divergences break the conformal symmetry (see ref. [18]¹). In the following we will focus on the finite parts. As a rule, the UV-divergent terms, after regularization, depend on the kinematics in relatively simple ways. Therefore, by focusing on the parts which are not UV divergent we capture the richest dependence on the kinematics.

After supersymmetrization a further complication appears: the right notion of light-like line is not one-dimensional anymore, but it has one bosonic dimension and several fermionic dimensions (see ref. [20]). In fact, the equations of motion of some gauge theories with extended supersymmetry are equivalent to the flatness of the field strength on such light-like lines. Despite these complications a super-Wilson loop can be constructed following the same steps as in ref. [21] and we do so in sec. 4.

In four dimensions, for $\mathcal{N} = 4$ super-Yang-Mills, a similar construction was done in chiral superspace in refs. [22, 23]. This construction is remarkable because of the duality with scattering amplitudes, which was uncovered in a series of papers [24–32].

Later, in ref. [33], Caron-Huot considered a *nonchiral* version of the super-Wilson loop, but he only considered it to the lowest order in the non-chiral variables $\bar{\theta}$. By computing the terms of type $\theta_i \bar{\theta}_j$ where the $\theta_i, \bar{\theta}_j$ correspond to different vertices² of the super-Wilson loop, he was able to compute the symbol (see [34]) of the two-loop planar n -sided bosonic Wilson loop.

It is our objective in this paper to render this procedure more systematic on the example of ABJM super-Wilson loops. Our focus is on the construction of null polygonal supersymmetric Wilson loops; however, in principle there should be no obstruction to defining a smooth supersymmetric Wilson loop, as done in ref. [13] for $\mathcal{N} = 4$ super Yang–Mills.

2 General discussion

We consider an $\mathcal{N} = 6$ Chern-Simons theory in three dimensions, coupled to matter, with gauge covariant derivatives:

$$\nabla_\mu = \partial_\mu + A_\mu, \quad \nabla_{IJ,\alpha} = D_{IJ,\alpha} + A_{IJ,\alpha}, \quad (2.1)$$

¹Dimensional regularization in Chern–Simons is subtle, see [19] for a detailed analysis.

²In fact, it turns out to be better to consider odd variables associated to the sides instead.

where α is a spinor index transforming in the two-dimensional spin representation of the three-dimensional Lorentz group $SO(1, 2) \simeq SL(2, \mathbb{R})$ and the indices I, J take four values and transform in the representation $\bar{\mathbf{4}}$ of $SU(4)$ when in the lower position and in $\mathbf{4}$ when in the upper position. The supersymmetry covariant derivative D_{IJ} is antisymmetric in the pair of indices IJ and is given by

$$D_{IJ,\alpha} = \frac{\partial}{\partial \theta^{IJ,\alpha}} + (\gamma^\mu \bar{\theta}_{IJ})_\alpha \partial_\mu, \quad (2.2)$$

where θ^{IJ} are the odd coordinates of $\mathcal{N} = 6$ superspace. Here the covariant derivatives and gauge connections transform as $(\mathbf{Ad}, \mathbf{Ad})$ under $U(N) \times U(M)$ gauge group. They can be thought of as $(N+M) \times (N+M)$ matrices with vanishing off-diagonal $N \times M$ and $M \times N$ blocks. The $\bar{\theta}$ are not independent on θ but are related by $\bar{\theta}_{IJ} = \frac{1}{2} \epsilon_{IJKL} \theta^{KL}$.

Using

$$\partial_{\alpha,IJ} \bar{\theta}_{KL}^\beta = \frac{1}{2} \epsilon_{KLMN} \partial_{\alpha,IJ} \theta^{\beta,MN} = \frac{1}{2} \epsilon_{KLMN} \delta_\alpha^\beta (\delta_I^M \delta_J^N - \delta_J^M \delta_I^N) = \epsilon_{IJKL} \delta_\alpha^\beta, \quad (2.3)$$

we find

$$\{D_{IJ,\alpha}, D_{KL,\beta}\} = 2\epsilon_{IJKL} \gamma_{\alpha\beta}^\mu \partial_\mu. \quad (2.4)$$

From now on we will use the notation $\partial_{\alpha\beta} \equiv \gamma_{\alpha\beta}^\mu \partial_\mu$. Under hermitian conjugation we have $\partial^\dagger = -\partial$, $D_{IJ}^\dagger = \frac{1}{2} \epsilon^{IJKL} D_{KL}$. The gauge covariant derivatives have the same hermitian conjugation properties as the supersymmetry covariant derivatives so we obtain $A_\mu^\dagger = -A_\mu$, $(A_{IJ})^\dagger = \frac{1}{2} \epsilon^{IJKL} A_{KL}$. We use conventions such that for two Grassmann variables ψ, χ we have $(\psi\chi)^\dagger = -\chi^\dagger \psi^\dagger$ and similarly for the transposition. This explains the absence of some factors of i which often appear in the literature on supersymmetry.

The analysis of the superspace constraints for the $\mathcal{N} = 6$ and $\mathcal{N} = 8$ super-Chern-Simons theories has been done in refs. [35, 36]. The gauge covariant derivatives satisfy the following algebra

$$\{\nabla_{IJ,\alpha}, \nabla_{KL,\beta}\} - 2\epsilon_{IJKL} \nabla_{\alpha\beta} = \epsilon_{\alpha\beta} \epsilon_{MIJ[K} W_{L]}^M \equiv F_{IJ\alpha, KL\beta}, \quad (2.5)$$

$$[\nabla_{IJ,\alpha}, \nabla_{\beta\gamma}] \equiv F_{IJ\alpha, \beta\gamma}, \quad (2.6)$$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] \equiv F_{\alpha\beta, \gamma\delta}. \quad (2.7)$$

In the first equation above we have already imposed some constraints (the discussion follows closely the refs. [35, 36] but with some change in conventions). Let us study the representations under the Lorentz $SL(2)$ and R -symmetry $SU(4)$ groups. The LHS transforms as $(\mathbf{2}, \mathbf{6}) \otimes_S (\mathbf{2}, \mathbf{6}) = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{15}) \oplus (\mathbf{3}, \mathbf{20}')$ under $SL(2) \times SU(4)$.³ We have already imposed the conventional constraint that the

³There are several conventions for naming the $SU(4)$ representations. Here we use the conven-

tions $\mathbf{20}' \simeq \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$, $\mathbf{20} \simeq \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$, $\bar{\mathbf{20}} \simeq \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$.

(**3**, **20'**) part vanishes. According to the analysis in refs. [35, 36] the inclusion of terms transforming as (**3**, **20'**) is only needed when studying higher derivative corrections. The traceless field W^I_J transforms as **15** of $SU(4)$. Under hermitian conjugation it transforms as $(W^I_J)^\dagger = W^J_I$. Later we will show that it is not an independent field, but it is a composite of the matter fields charged under the Chern-Simons gauge group. Another way to write the tensor W is as $W_{IJ,KL}$, which is antisymmetric in (I, J) , (K, L) and in the exchange of the pairs (I, J) and (K, L) . The relation to the previous form is given by $W^P_L = -\frac{1}{2}\epsilon^{IJKP}W_{IJ,KL}$.

The Bianchi identity involving three covariant derivatives $\nabla_{IJ\alpha}$, $\nabla_{KL\beta}$ and $\nabla_{MN\gamma}$ yields

$$\epsilon_{KLMN}F_{IJ\alpha,\beta\gamma} + \frac{1}{2}\epsilon_{\beta\gamma}\epsilon_{PKL[M]}\nabla_{IJ\alpha}[W^P_{|N|}] + \text{cyclic}\{(IJ\alpha), (KL\beta), (MN\gamma)\} = 0. \quad (2.8)$$

This imposes some constraints on the derivatives of the W superfield. In order to extract these constraints let us contract this equation with ϵ^{KLMN} . After the contraction we obtain

$$24F_{IJ\alpha,\beta\gamma} + 4F_{IJ\gamma,\alpha\beta} + 4F_{IJ\beta,\gamma\alpha} + \epsilon_{\alpha\beta} \times (\cdots)_\gamma + \epsilon_{\beta\gamma} \times (\cdots)_\alpha + \epsilon_{\gamma\alpha} \times (\cdots)_\beta = 0, \quad (2.9)$$

where we have indicated some of the terms only schematically. This implies that the completely symmetric part (or spin $\frac{3}{2}$ under $SL(2)$) vanishes separately, that is

$$F_{IJ\alpha,\beta\gamma} + F_{IJ\gamma,\alpha\beta} + F_{IJ\beta,\gamma\alpha} = 0, \quad (2.10)$$

which can be solved as

$$F_{IJ\alpha,\beta\gamma} = \frac{1}{3}(\epsilon_{\alpha\beta}F_{IJ,\gamma\delta}^\delta + \epsilon_{\alpha\gamma}F_{IJ,\beta\delta}^\delta). \quad (2.11)$$

If after contracting with ϵ^{KLMN} we further contract with $\epsilon^{\alpha\beta}$, we finally obtain

$$F_{IJ}{}^\alpha{}_{\gamma\alpha} = \frac{3}{5}[\nabla_{P[I|\gamma}, W^P_{|J|}]. \quad (2.12)$$

From this one can reconstruct the field strength $F_{IJ\alpha,\beta\gamma}$. Plugging back into the Bianchi identity we find a constraint uniquely between the derivatives $\nabla_{IJ}W^K_L$. These quantities transform as $\mathbf{6} \otimes \mathbf{15} = \mathbf{6} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{64}$ under $SU(4)$. The constraint mentioned above forces the **64** part to vanish. Therefore, the derivative of W has a special form

$$[\nabla_{IJ\alpha}, W^K_L] = \delta_{[I}^K\lambda_{\alpha|J]L} + \frac{1}{4}\delta_L^K\lambda_{\alpha IJ} + \delta_{[I}^K\rho_{\alpha|J]L} - \frac{1}{2}\epsilon_{IJLN}\bar{\rho}_\alpha^{KN}, \quad (2.13)$$

where λ_{IJ} transforms in the **6** of $SU(4)$ while ρ_{IJ} , $\bar{\rho}^{IJ}$ transform in the **10** and $\overline{\mathbf{10}}$ respectively. The form for the derivative of W is fixed by the $SU(4)$ transformations, the tracelessness of W and reality conditions.

Using the special form of $[\nabla, W]$ we can easily compute

$$[\nabla_{P[I|\alpha}, W^P_{|J]}] = \frac{5}{4}\lambda_{\alpha IJ}, \quad [\nabla_{P(I|\alpha}, W^P_{|J)}] = \frac{3}{2}\rho_{\alpha IJ}. \quad (2.14)$$

Also,

$$F_{IJ\alpha, \beta\gamma} = \frac{1}{4}\epsilon_{\alpha\beta}\lambda_{\gamma IJ} + \frac{1}{4}\epsilon_{\alpha\gamma}\lambda_{\beta IJ}. \quad (2.15)$$

The second Bianchi identity, between derivatives $\nabla_{\alpha\beta}$, $\nabla_{IJ\gamma}$ and $\nabla_{KL\delta}$ gives the field strength $F_{\alpha\beta} \equiv \epsilon^{\gamma\delta}F_{\alpha\gamma, \beta\delta}$ in terms of covariant derivatives of W . This is in fact the equation of motion for F and we will return to it later, after we express W in terms of fundamental fields.

It can be shown that the remaining Bianchi identities do not impose extra conditions. Therefore, we have finished the analysis of the gauge sector and we now move on to analyzing the matter sector. The matter fields are organized in two kinds of superfields: scalar Φ^I transforming in **4** of $SU(4)$ and $\Psi_{I\alpha}$ transforming in $\bar{\mathbf{4}}$. We take these matter superfields to transform as $(\mathbf{N}, \bar{\mathbf{M}})$ under the gauge group $U(N) \times U(M)$. The hermitian conjugate fields $\bar{\Phi}_I = (\Phi^I)^\dagger$, $\bar{\Psi}^I = (\Psi_I)^\dagger$ transform as $(\mathbf{M}, \bar{\mathbf{N}})$. In terms of algebra generators, the gauge fields have components in the diagonal blocks while the matter fields have components in the off-diagonal blocks.

The derivative $[\nabla_{IJ\alpha}, \Phi^K]$ transforms in $\mathbf{4} \otimes \mathbf{6} = \bar{\mathbf{4}} \oplus \mathbf{20}$. We impose the constraint that $\mathbf{20}$ component vanishes. In turn, this constrains the form of W as we will show. We take

$$[\nabla_{IJ\alpha}, \Phi^K] = \delta_{|I}^K \Psi_{|J]\alpha}. \quad (2.16)$$

If we apply to this equation $\nabla_{KL\beta}$ and we use the anticommutation relations derived previously, we find

$$\delta_{[I}^P \{\nabla_{KL\beta}, \Psi_{|J]\alpha}\} + (IJ\alpha) \leftrightarrow (KL\beta) = 2\epsilon_{IJKL}[\nabla_{\alpha\beta}, \Phi^P] + \epsilon_{\alpha\beta}\epsilon_{MIJ[K}[\nabla_{|L]}, \Phi^P]. \quad (2.17)$$

If we contract with $\epsilon^{\alpha\beta}\epsilon^{IJKN}$ we find the constraint

$$[W_{|J}^{(I}, \Phi^{K)}] = \frac{1}{5}\delta_J^{(I} [W_{|N}^{K)}, \Phi^N]. \quad (2.18)$$

This is equivalent to the statement that the **36** in the decomposition $\mathbf{15} \otimes \mathbf{4} = \mathbf{4} \oplus \mathbf{20} \oplus \mathbf{36}$ of $[W_{|J}^I, \Phi^K]$ is set to zero.

The constraint in eq. (2.17) relates the fermionic derivative of the spinor superfield to the derivative of the scalar superfield and the commutator $[W, \Phi]$. In order to find the content of this constraint, we project the LHS and RHS onto the irreps $(\mathbf{4}, \mathbf{1})$, $(\mathbf{4}, \mathbf{3})$, $(\mathbf{20}, \mathbf{1})$ and $(\mathbf{20}, \mathbf{3})$. Combining these projections with appropriate coefficients we find

$$\{\nabla_{IJ\alpha}, \Psi_{K\beta}\} = -4\epsilon_{IJKL}\nabla_{\alpha\beta}\Phi^L + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{MIJL}[W_{|K}^M, \Phi^L] - \frac{3}{10}\epsilon_{\alpha\beta}\epsilon_{MIJK}[W_{|L}^M, \Phi^L]. \quad (2.19)$$

Now we want to express W in terms of the fields in our theory. Given the dimension of W , its R -symmetry transformations, its tracelessness, its hermiticity properties and the constraint in eq. (2.18), the answer is heavily constrained. The choice which yields the ABJ theory [37] is

$$W^I_J = \frac{1}{g} \left(\Phi^I \bar{\Phi}_J - \frac{1}{4} \delta^I_J \Phi^K \bar{\Phi}_K + \bar{\Phi}_J \Phi^I - \frac{1}{4} \delta^I_J \bar{\Phi}_K \Phi^K \right). \quad (2.20)$$

The first two terms in W transform as the adjoint of $U(N)$ while the last two transform as the adjoint of $U(M)$. In order to obtain the ABJM theory we set $N = M$. Also, the coupling constant g is quantized such that the exponential of the action e^{iS} is gauge invariant. We will fix the coupling constant later when we discuss the Lagrangian of the theory.

Let us now write the equations of motion of the theory. The equation of motion for the field strength is obtained from the Bianchi identity of $\nabla_{\alpha\beta}$, $\nabla_{IJ\gamma}$ and $\nabla_{KL\delta}$. Before we write the equation of motion we should recall that we have two gauge groups: $U(N)$ with gauge field A and $U(M)$ with gauge field \tilde{A} . Therefore, we have two field strengths, F and \tilde{F} . The equations of motion for the gauge fields are:

$$F_{\alpha\gamma} = \frac{1}{4g} \left([\nabla_{\alpha\gamma}, \Phi^I] \bar{\Phi}_I - \Phi^I [\nabla_{\alpha\gamma}, \bar{\Phi}_I] - \frac{1}{4} \Psi_{I(\alpha} \bar{\Psi}_{\gamma)}^I \right), \quad (2.21)$$

$$\tilde{F}_{\alpha\gamma} = -\frac{1}{4g} \left([\nabla_{\alpha\gamma}, \bar{\Phi}_I] \Phi^I - \bar{\Phi}_I [\nabla_{\alpha\gamma}, \Phi^I] - \frac{1}{4} \bar{\Psi}_{(\alpha}^I \Psi_{I\gamma)} \right). \quad (2.22)$$

The equations of motion for the fermions are obtained from the Bianchi identity for $\nabla_{IJ\alpha}$, $\nabla_{KL\beta}$ and $\Psi_{M\gamma}$. We obtain

$$\epsilon^{\beta\gamma} [\nabla_{\alpha\beta}, \Psi_{I\gamma}] = \frac{\Phi^J \bar{\Phi}_I \Psi_{J\alpha}}{4g} - \frac{\Phi^J \bar{\Phi}_J \Psi_{I\alpha}}{8g} + \frac{\Psi_{I\alpha} \bar{\Phi}_J \Phi^J}{8g} - \frac{\Psi_{J\alpha} \bar{\Phi}_I \Phi^J}{4g} + \frac{\epsilon_{IJKL} (\Phi^J \bar{\Psi}_{\alpha}^K \Phi^L)}{4g}. \quad (2.23)$$

Finally, the equation of motion for the scalars can be determined from the equation of motion for the fermions, by taking the anticommutator of the fermion equation of motion with $\nabla_{LP\gamma}$, multiplying by $\epsilon^{\alpha\gamma} \epsilon^{LMPQ}$ and using the Jacobi identities and the constraints. We obtain

$$\begin{aligned} \epsilon^{\alpha\beta'} \epsilon^{\alpha'\beta} [\nabla_{\alpha\alpha'}, [\nabla_{\beta\beta'}, \Phi^Q]] &= \frac{\epsilon^{\alpha\alpha'} (\Phi^I \bar{\Psi}_{\alpha}^Q \Psi_{I\alpha'})}{16g} - \frac{\epsilon^{\alpha\alpha'} (\Phi^Q \bar{\Psi}_{\alpha}^I \Psi_{I\alpha'})}{32g} + \\ &\quad \frac{\epsilon^{\alpha\alpha'} \epsilon^{IJKQ} (\Psi_{I\alpha} \bar{\Phi}_J \Psi_{K\alpha'})}{16g} + \frac{\epsilon^{\alpha\alpha'} (\Psi_{I\alpha} \bar{\Psi}_{\alpha'}^I \Phi^Q)}{32g} - \frac{\epsilon^{\alpha\alpha'} (\Psi_{I\alpha} \bar{\Psi}_{\alpha'}^Q \Phi^I)}{16g} + \\ &\quad \frac{\Phi^I \bar{\Phi}_I \Phi^J \bar{\Phi}_J \Phi^Q}{32g^2} - \frac{\Phi^I \bar{\Phi}_I \Phi^Q \bar{\Phi}_J \Phi^J}{16g^2} - \frac{\Phi^I \bar{\Phi}_J \Phi^J \bar{\Phi}_I \Phi^Q}{16g^2} + \frac{\Phi^I \bar{\Phi}_J \Phi^Q \bar{\Phi}_I \Phi^J}{8g^2} + \\ &\quad \frac{\Phi^Q \bar{\Phi}_I \Phi^I \bar{\Phi}_J \Phi^J}{32g^2} - \frac{\Phi^Q \bar{\Phi}_I \Phi^J \bar{\Phi}_J \Phi^I}{16g^2}. \end{aligned} \quad (2.24)$$

These equations of motion can be obtained from the Lagrangian⁴

$$\begin{aligned} \mathcal{L}_{\text{ABJM}} = & -\text{tr}([\nabla_\mu, \Phi^I][\nabla^\mu, \bar{\Phi}_I]) + \frac{i}{8}\text{tr}(\bar{\Psi}^{\alpha I}[\nabla_{\alpha\beta}, \Psi_I^\beta]) - \\ & 4g\epsilon^{\mu\nu\lambda}\text{tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda - \tilde{A}_\mu\partial_\nu\tilde{A}_\lambda - \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\lambda\right) - V_{\text{bos}} + \mathcal{L}_{\text{Yuk}}, \end{aligned} \quad (2.25)$$

where

$$\begin{aligned} V_{\text{bos}} = & \frac{1}{192g^2}\text{tr}\left(\Phi^I\bar{\Phi}_I\Phi^J\bar{\Phi}_J\Phi^K\bar{\Phi}_K + \Phi^I\bar{\Phi}_J\Phi^J\bar{\Phi}_K\Phi^K\bar{\Phi}_I + \right. \\ & \left. 4\Phi^I\bar{\Phi}_J\Phi^K\bar{\Phi}_I\Phi^J\bar{\Phi}_K - 6\Phi^I\bar{\Phi}_K\Phi^J\bar{\Phi}_J\Phi^K\bar{\Phi}_I\right), \\ \mathcal{L}_{\text{Yuk}} = & -\frac{\epsilon^{\alpha\beta}}{64g}\text{tr}\left(-\epsilon^{IJKL}\bar{\Phi}_I\Psi_{J\alpha}\bar{\Phi}_K\Psi_{L\beta} + \epsilon_{IJKL}\Phi^I\bar{\Psi}_\alpha^J\Phi^K\bar{\Psi}_\beta^L + \right. \\ & \left. 2\bar{\Phi}_I\Phi^J\bar{\Psi}_\alpha^I\Psi_{J\beta} + \Phi^I\bar{\Phi}_I\Psi_{J\alpha}\bar{\Psi}_\beta^J - 2\Phi^I\bar{\Phi}_J\Psi_{I\alpha}\bar{\Psi}_\beta^J - \bar{\Phi}_I\Phi^I\bar{\Psi}_\alpha^J\Psi_{J\beta}\right). \end{aligned}$$

We could rescale the fields $\Phi \rightarrow g^{1/2}\Phi$ and $\Psi \rightarrow g^{1/2}\Psi$ to pull the coupling constant g in front of the Lagrangian.

The gauge part of the action can also be written in spinor language as

$$\frac{1}{2}\epsilon^{\alpha\alpha'}\epsilon^{\beta\delta}\epsilon^{\gamma\beta'}A_{\alpha'\beta'}\left(\partial_{\alpha\beta}A_{\gamma\delta} + \frac{2}{3}A_{\alpha\beta}A_{\gamma\delta}\right) = \epsilon^{\mu\nu\lambda}\text{tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right). \quad (2.26)$$

The first form is closer to the language we've been using so far, but the second form is used to find the quantization condition on the coupling. The quantization of the Chern-Simons coupling is a consequence of the gauge non-invariance of the Lagrangian. It can be checked that under a gauge transformation $A_\mu \rightarrow A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g$ we have

$$\begin{aligned} \delta_{\text{gauge}}\left(\epsilon^{\mu\nu\rho}\text{tr}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)\right) = \\ \partial_\rho\left(\epsilon^{\mu\nu\rho}\text{tr}(A_\mu\partial_\nu gg^{-1})\right) - \frac{1}{3}\epsilon^{\mu\nu\rho}\text{tr}(g^{-1}\partial_\mu gg^{-1}\partial_\nu gg^{-1}\partial_\rho g). \end{aligned} \quad (2.27)$$

The first term is a total derivative which we will ignore. The integral of the second term is quantized (it is an integer multiple of $8\pi^2$). In order for the exponential of the action to be gauge invariant, we need to choose a global coupling of $\frac{k}{4\pi}$, where k is an integer (at the same time, we need to pick an appropriate normalization⁵ of the gauge generators, which for $SU(N)$ is $\text{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}$).

We should mention that these properties of the Chern-Simons Lagrangian distinguish it from the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian in four dimensions. For

⁴Actually this Lagrangian is a superfield whose lowest component is the actual Lagrangian.

⁵We choose the generators of the gauge algebra T^a to be antihermitian. This choice eliminates some factors of i from the action.

$\mathcal{N} = 4$ super-Yang-Mills, the on-shell Lagrangian can be written as a descendant of a protected operator (see ref. [38]). Because of the gauge non-invariance, this can not work here. Presumably this makes it more challenging to use supersymmetry to simplify perturbative computations as has been done for example in ref. [39]. In planar $\mathcal{N} = 4$ super-Yang-Mills, polygonal light-like Wilson loops can also be computed by taking limits of correlation functions of BPS operators, as has been shown in ref. [40]; the analogous analysis for ABJM has been studied in ref. [41].

Now we can work out the components of the gauge connections. In order to eliminate the auxiliary fields we use the Harnad & Shnider gauge (see ref. [42]) $\theta^{IJ\alpha} A_{IJ\alpha} = 0$. Using this gauge condition and the symmetry of the gamma matrices have that $D \equiv \frac{1}{2}\theta^{IJ\alpha}\nabla_{IJ\alpha} = \frac{1}{2}\theta^{IJ\alpha}\partial_{IJ\alpha}$. Then, we obtain

$$(1 + D)A_{KL\beta} = 2\bar{\theta}_{KL}^{\alpha}A_{\alpha\beta} + \epsilon_{\alpha\beta}\bar{\theta}_{M[K}^{\alpha}W^M_{L]}, \quad (2.28)$$

$$DA_{\beta\gamma} = -\frac{1}{4}\theta_{(\beta}^{IJ}\lambda_{IJ,\gamma)}, \quad (2.29)$$

$$D\Phi^K = \frac{1}{2}\theta^{KL\alpha}\Psi_{L\alpha}, \quad (2.30)$$

$$D\bar{\Phi}_K = \frac{1}{2}\bar{\theta}_{KL}^{\alpha}\bar{\Psi}_{\alpha}^L, \quad (2.31)$$

$$D\Psi_{K\beta} = -4\bar{\theta}_{KL}^{\alpha}[\nabla_{\alpha\beta}, \Phi^L] - \frac{1}{2}\bar{\theta}_{ML\beta}[W^M_K, \Phi^L] + \frac{3}{10}\bar{\theta}_{MK\beta}[W^M_L, \Phi^L], \quad (2.32)$$

$$D\bar{\Psi}_{\beta}^K = 4\theta^{IK\alpha}[\nabla_{\alpha\beta}, \bar{\Phi}_I] + \frac{1}{2}\theta_{\beta}^{IJ}[\bar{\Phi}_I, W^K_J] + \frac{3}{10}\theta_{\beta}^{IK}[\bar{\Phi}_J, W^J_I]. \quad (2.33)$$

Now we can build the components of the connection in superspace by using the D -recursion relations and the initial conditions $A_{\alpha\beta}| = a_{\alpha\beta}$, $\Phi^I| = \phi^I$, $\Psi_{I\alpha}| = \psi_{I\alpha}$, $A_{IJ\alpha}| = 0$. The θ expansion to the first few orders is

$$A_{KL\beta} = \bar{\theta}_{KL}^{\alpha}a_{\alpha\beta} + \frac{1}{8g}\epsilon_{\beta\alpha}\bar{\theta}_{KL}^{\alpha}\phi^P\bar{\phi}_P + \frac{1}{2g}\epsilon_{\beta\alpha}\bar{\theta}_{M[K}^{\alpha}\phi^M\bar{\phi}_{L]} + \dots, \quad (2.34)$$

$$A_{\beta\gamma} = a_{\beta\gamma} - \frac{1}{4g}(\theta_{(\beta}^{IJ}\psi_{I|\gamma)}\bar{\phi}_J + \bar{\theta}_{IJ(\beta}\phi^I\bar{\psi}_{\gamma)}^J) + \dots, \quad (2.35)$$

where we have only written the part of the gauge connection transforming in the adjoint of $U(N)$; the part transforming in the adjoint of $U(M)$ is similar.

Let us list the form of the field strengths

$$F_{IJ\alpha,KL\beta} = \epsilon_{\alpha\beta}\epsilon_{MIJ[K}W^M_{L]}, \quad (2.36)$$

$$F_{IJ\alpha,\beta\gamma} = \frac{1}{3}(\epsilon_{\alpha\beta}F_{IJ,\gamma\delta}^{\delta} + \epsilon_{\alpha\gamma}F_{IJ,\beta\delta}^{\delta}), \quad (2.37)$$

$$F_{\alpha\beta,\gamma\delta} = -\frac{1}{4}(\epsilon_{\alpha\gamma}F_{\beta\delta} + \epsilon_{\beta\delta}F_{\alpha\gamma} + \epsilon_{\beta\gamma}F_{\alpha\delta} + \epsilon_{\alpha\delta}F_{\beta\gamma}). \quad (2.38)$$

It is easy to see that if $\delta x_{\alpha\beta} = t\lambda_{\alpha}\lambda_{\beta}$ and $\delta\theta^{IJ,\alpha} = \lambda^{\alpha}\eta^{IJ}$, then the field strengths vanish when contracted with δx and $\delta\theta$. What about the converse statement? That

is, given the flatness conditions on such submanifolds, do they imply the constraints and the equations of motion? This is not true. One missing constraint is the one in eq. (2.18), which arises in the matter sector. Another missing constraint is the constraint that W transforms in **15** of $SU(4)$.

It would be interesting to look for a formulation where all the equations of motion arise from flatness conditions on some submanifolds of superspace which are well-behaved under superconformal transformations. A way to search for such a formulation is to look at the flag manifolds of $\mathbb{C}^{4|6}$ which is a space with a natural action of the superconformal group $OSp(6|4)$ (see refs. [43, 44] for the description of the general theory). A promising choice seems to be the flag $\mathbb{C}^{2|3} \subset \mathbb{C}^{4|6}$ which is also distinguished by the fact that it does not have supersymmetric torsion, just like chiral superspaces.

Such a formulation, if it exists, would be the basis of a twistorial formulation of ABJM theory and would probably be a good starting point for studying the dual of ABJM scattering amplitudes. The $\mathcal{N} = 4$ super-Yang-Mills scattering amplitudes enjoy a Yangian symmetry (see ref. [45]) and the same holds for the ABJM scattering amplitudes (see ref. [46, 47]). However, at strong coupling the fate of the Yangian symmetry is less clear [48, 49]. See also ref. [50] for a study of the Wilson-loop/scattering amplitudes duality in ABJM theory at four points, where the tree-level scattering amplitude is factored out.

3 Superwistors for $OSp(6|4)$

In this section we introduce $OSp(6|4)$ supertwistors. We take $x_{\alpha\beta}$ to be a 2×2 matrix and θ_{α}^{IJ} to be a 2×6 matrix. When we write products of x and θ the matrix product is understood. The contraction $\theta\varpi\theta^T$ is defined with the help of the $SU(4)$ ϵ tensor: $(\theta\varpi\theta^T)_{\alpha\beta} = \frac{1}{4}\theta_{\alpha}^{IJ}\epsilon_{IJKL}\theta_{\beta}^{KL}$. As a rule, whenever we contract two antisymmetric indices we include a factor of $\frac{1}{2}$ to prevent double counting. For example, $\varpi\theta$ is a shorthand notation for $\frac{1}{2}\epsilon_{IJKL}\theta^{KL}$. Notice that the 2×2 matrix $\theta\varpi\theta^T$ is antisymmetric.

We define $x^{\pm} = x \pm \frac{1}{2}\theta\varpi\theta^T$. Then, given (x^{\pm}, θ) we define two two-planes in $\mathbb{C}^{4|6}$ by

$$\lambda \mapsto (\lambda, \lambda x^+, \lambda\theta) \equiv (\lambda, \mu, \chi) = Z, \quad (3.1)$$

$$\lambda \mapsto \begin{pmatrix} -x^-\lambda^T \\ \lambda^T \\ -\varpi\theta^T\lambda^T \end{pmatrix} \equiv \begin{pmatrix} \tilde{\mu} \\ \lambda^T \\ \tilde{\chi} \end{pmatrix} = \bar{Z}. \quad (3.2)$$

More properly, the second two-plane lives in the dual $\mathbb{C}^{4|6}$. Then we have

$$Z \cdot \bar{Z} = \lambda(x^+ - x^- - \theta\varpi\theta^T)\lambda^T = 0, \quad (3.3)$$

which implies that the two two-planes are orthogonal. Said differently, the contraction is performed with a symplectic form which is antisymmetric. Under a superconformal transformation $h \in OSp(6|4)$ we have $Z \rightarrow Zh$ and $\bar{Z} \rightarrow h^{-1}\bar{Z}$. Therefore, simple superconformal invariants can be obtained by contraction $Z \cdot \bar{Z}'$.

A choice of Z, \bar{Z} such that $Z \cdot \bar{Z} = 0$ yields a light-like line. If x_0^\pm, θ_0 is a particular solution of the twistor equations such that $x_0^+ - x_0^- = \theta_0 \varpi \theta_0^T$, then the general solution is:

$$x^+ = x_0^+ + \epsilon \lambda^T \eta \varpi \theta_0^T + t \epsilon \lambda^T \lambda \epsilon, \quad (3.4)$$

$$x^- = x_0^- + \theta_0 \varpi \eta^T \lambda \epsilon + t \epsilon \lambda^T \lambda \epsilon, \quad (3.5)$$

$$\theta = \theta_0 + \epsilon \lambda^T \eta, \quad \theta^T = \theta_0^T - \eta^T \lambda \epsilon, \quad (3.6)$$

$$x = x_0 + t \epsilon \lambda^T \lambda \epsilon + \frac{1}{2} (\epsilon \lambda^T \eta \varpi \theta_0^T + \theta_0 \varpi \eta^T \lambda \epsilon), \quad (3.7)$$

where ϵ is a 2×2 antisymmetric matrix such that $\epsilon^2 = -1$. Therefore, we see that a light-like line has dimension $(1|6)$. The translations along this “fat” line are generated by the following vector fields

$$\epsilon \lambda^T \lambda \epsilon \frac{\partial}{\partial x}, \quad (3.8)$$

$$\epsilon \lambda^T \eta \frac{\partial}{\partial \theta} + \frac{1}{2} \epsilon \lambda^T \eta \varpi \theta^T \frac{\partial}{\partial x} + \frac{1}{2} \theta \varpi \eta^T \lambda \epsilon \frac{\partial}{\partial x} = \epsilon \lambda^T \eta \left(\frac{\partial}{\partial \theta} + \varpi \theta^T \frac{\partial}{\partial x} \right), \quad (3.9)$$

where we have used the fact that x is symmetric as a matrix. Above, we recognize the SUSY covariant derivative D .

Let us introduce the bosonic and fermionic vielbeine. The total derivative in variables (x, θ) is given by

$$d = d\theta \cdot \frac{\partial}{\partial \theta} + dx \cdot \frac{\partial}{\partial x}, \quad (3.10)$$

where \cdot stands for total contraction of indices. This total derivative can also be written as

$$d = e_B \cdot \frac{\partial}{\partial x} + e_F \cdot D, \quad (3.11)$$

where D is the fermionic SUSY covariant derivative. Identifying terms we obtain

$$e_B = dx - \frac{1}{2} d\theta \varpi \theta^T + \frac{1}{2} \theta \varpi d\theta^T, \quad e_F = d\theta, \quad (3.12)$$

where we have symmetrized e_B since it is contracted with a symmetric tensor.

Two light-like lines given by (Z, \bar{Z}) and (Z', \bar{Z}') respectively, intersect if $Z \cdot \bar{Z}' = 0$ and $Z' \cdot \bar{Z} = 0$. We also have $Z \cdot \bar{Z} = 0$ and $Z' \cdot \bar{Z}' = 0$, as explained above.

The considerations above also lead to a natural definition of supersymmetry invariant distance between points (x_i, θ_i) and (x_j, θ_j)

$$\Delta x_{ij} = x_i^+ - x_j^- - \theta_i \varpi \theta_j^T, \quad (3.13)$$

which is such that $Z_i \cdot \bar{Z}_j = \lambda_i \Delta x_{ij} \lambda_j^T$. Taking into account that $(\Delta x_{ij})^T = -\Delta x_{ji}$, we find that $Z_i \cdot \bar{Z}_j = -Z_j \cdot \bar{Z}_i$. There are other ways to define intervals which are invariant under super-Poincaré transformations. They all differ by nilpotent terms. The Δx_{ij} defined above is the three-dimensional counterpart of a chiral–antichiral interval. It has the peculiarity that $\Delta x_{ij} \neq -\Delta x_{ji}$.

We can define another super-Poincaré invariant interval δx_{ij} which is such that $\delta x_{ij} = -\delta x_{ji}$ by taking

$$\delta x_{ij} = \frac{1}{2}(\Delta x_{ij} - \Delta x_{ji}) = x_i - x_j - \frac{1}{2}\theta_i \varpi \theta_j^T + \frac{1}{2}\theta_j \varpi \theta_i^T. \quad (3.14)$$

We also have $\delta x_{ij}^T = \delta x_{ij}$.

The configuration of pairwise light-like separated points in superspace can be given as a sequence of twistors Z_i, \bar{Z}_i such that $Z_i \cdot \bar{Z}_i = 0$, $Z_i \cdot \bar{Z}_{i\pm 1} = 0$. We solve for x_i, θ_i from the components of Z_i and \bar{Z}_i . Using

$$\mu_i = \lambda_i x_i^+, \quad \mu_{i+1} = \lambda_{i+1} x_i^+, \quad (3.15)$$

$$\tilde{\mu}_i = -x_i^- \lambda_i^T, \quad \tilde{\mu}_{i+1} = -x_i^- \lambda_{i+1}^T, \quad (3.16)$$

$$\chi_i = \lambda_i \theta_i, \quad \chi_{i+1} = \lambda_{i+1} \theta_i, \quad (3.17)$$

we find

$$x_i^+ = \frac{\epsilon \lambda_{i+1}^T \mu_i - \epsilon \lambda_i^T \mu_{i+1}}{\lambda_i \epsilon \lambda_{i+1}^T}, \quad (3.18)$$

$$x_i^- = \frac{\tilde{\mu}_i \lambda_{i+1} \epsilon - \tilde{\mu}_{i+1} \lambda_i \epsilon}{\lambda_i \epsilon \lambda_{i+1}^T}, \quad (3.19)$$

$$\theta_i = \frac{\epsilon \lambda_{i+1}^T \chi_i - \epsilon \lambda_i^T \chi_{i+1}}{\lambda_i \epsilon \lambda_{i+1}^T}. \quad (3.20)$$

Geometrically, these equations describe the point (x_i, θ_i) as the intersection of two light-like lines represented by (Z_i, \bar{Z}_i) and (Z_{i+1}, \bar{Z}_{i+1}) , respectively.

It is easy to show that

$$\theta_i - \theta_{i+1} = \epsilon \lambda_{i+1}^T \eta_{i+1}, \quad (3.21)$$

where

$$\eta_i = \frac{\langle i-1i \rangle \chi_{i+1} + \langle ii+1 \rangle \chi_{i-1} + \langle i+1i-1 \rangle \chi_i}{\langle i-1i \rangle \langle ii+1 \rangle} \quad (3.22)$$

and $\langle ij \rangle = \lambda_i \epsilon \lambda_j^T$.

In order to compute the distance squared $(\Delta x_{ij})^2$ we first compute the matrix elements

$$\lambda_i \Delta x_{ij} \lambda_j^T = Z_i \cdot \bar{Z}_j, \quad \lambda_i \Delta x_{ij} \lambda_{j+1}^T = Z_i \cdot \bar{Z}_{j+1}, \quad (3.23)$$

$$\lambda_{i+1} \Delta x_{ij} \lambda_j^T = Z_{i+1} \cdot \bar{Z}_j, \quad \lambda_{i+1} \Delta x_{ij} \lambda_{j+1}^T = Z_{i+1} \cdot \bar{Z}_{j+1}. \quad (3.24)$$

From this we find that

$$-(\Delta x_{ij})^2 = \det \Delta x_{ij} = \frac{(Z_i \cdot \bar{Z}_j)(Z_{i+1} \cdot \bar{Z}_{j+1}) - (Z_i \cdot \bar{Z}_{j+1})(Z_{i+1} \cdot \bar{Z}_j)}{\langle ii+1 \rangle \langle jj+1 \rangle}. \quad (3.25)$$

Above we mentioned that $\Delta x_{ij} \neq -\Delta x_{ji}$. However, for a light-like interval $\Delta x_{i,i+1} = \Delta x_{i+1,i}$ due to $(\theta_i - \theta_{i+1})\varpi(\theta_i - \theta_{i+1})^T = 0$. As a consequence $\Delta x_{i,i+1} = \delta x_{i,i+1}$.

Not all the quantities we will encounter can be expressed only in terms of twistors since they are not conformal invariant. For writing down quantities which are Lorentz but not conformal invariant we introduce an “infinity” twistor \mathcal{I} , which is such that $Z_i \mathcal{I} \bar{Z}_j = \langle ij \rangle$, $\text{str}(X_i \mathcal{I}) = -2\langle ii+1 \rangle$. In terms of $(4|6) \times (4|6)$ matrices the infinity twistor is

$$\mathcal{I} = \begin{pmatrix} 0 & \epsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.26)$$

This infinity twistor is also preserved (that is commutes with) the Poincaré supersymmetry. Conformal inversion acts as $X \rightarrow \Upsilon X \Upsilon$, where

$$\Upsilon = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.27)$$

The zero twistor, which is obtained by setting $x = 0$, $\theta = 0$ is obtained from the infinity twistor \mathcal{I} by inversion.

We are also interested in parametrizing the light-like segment between points with coordinates (x_i, θ_i) and (x_{i+1}, θ_{i+1}) . A point represented by the two pairs of twistors (Z_{i+1}, \bar{Z}_{i+1}) and $(\alpha Z_i + \beta Z_{i+2}, \alpha \bar{Z}_i + \beta \bar{Z}_{i+2})$ with $\alpha = (1 - \tau)\langle i+2, i+1 \rangle$, $\beta = \tau\langle ii+1 \rangle$ lies on the segment $(i, i+1)$. First of all, notice that these two pairs of twistors parametrize intersecting light-like lines since $Z_{i+1} \cdot (\alpha Z_i + \beta Z_{i+2}) = 0$. Then, solving for x^\pm, θ we find $x^\pm = (1 - \tau)x_i^\pm + \tau x_{i+1}^\pm$, $\theta = (1 - \tau)\theta_i + \tau\theta_{i+1}$ which also implies that $x = (1 - \tau)x_i + \tau x_{i+1}$. Then we compute the vielbeine along this curve and we obtain⁶

$$e_B = d\tau(x_{i+1} - x_i - \frac{1}{2}\theta_{i+1}\varpi\theta_i^T + \frac{1}{2}\theta_i\varpi\theta_{i+1}^T) = d\tau\delta x_{i+1,i} = d\tau\epsilon\lambda_{i+1}^T\lambda_{i+1}\epsilon, \quad (3.28)$$

$$e_F = d\tau(\theta_{i+1} - \theta_i) = d\tau\epsilon\lambda_{i+1}^T\eta_{i+1}. \quad (3.29)$$

There is an obvious ambiguity in parametrizing a point x_i as the intersection of two light-like lines since one can choose some other light-like lines with the same

⁶The formula for e_B implicitly contains a choice of normalization for λ_{i+1} . The expression which is invariant under rescaling is given by $e_B = \frac{(\mathcal{I}\bar{Z}_{i+1})(Z_{i+1}\mathcal{I})}{\langle ii+1\mathcal{I} \rangle \langle i+1i+2\mathcal{I} \rangle} (Z_i \cdot \bar{Z}_{i+2})$, where we have written all the infinity twistors explicitly $\langle ii+1\mathcal{I} \rangle = \langle ii+1 \rangle$ to show that e_B is invariant also under the rescaling of the infinity twistor. In the following we will assume that the λ_i have been normalized such that the bosonic vielbein on side i is given by $e_B = \epsilon\lambda_{i+1}^T\lambda_{i+1}\epsilon$.

intersection. Instead of using the lines determined by (Z_i, \bar{Z}_i) and (Z_{i+1}, \bar{Z}_{i+1}) , we can choose the lines determined by (Z'_i, \bar{Z}'_i) and $(Z'_{i+1}, \bar{Z}'_{i+1})$, where the primed quantities are determined from the unprimed ones by a 2×2 matrix. However, the quantity⁷

$$X_i = \bar{Z}_i \otimes Z_{i+1} - \bar{Z}_{i+1} \otimes Z_i \quad (3.30)$$

remains unchanged up to a rescaling by the determinant of the 2×2 transformation matrix. In order to make quantities which are independent on the scaling we have to make sure that these determinant factors cancel. For example, the ratio $X_i / \langle ii + 1 \rangle$ is invariant under rescaling.

Using the fact that $\lambda_i^T \otimes \lambda_{i+1} - \lambda_{i+1}^T \otimes \lambda_i = \epsilon \langle ii + 1 \rangle$, which can be obtained by considering contractions with $\epsilon \lambda_i^T$ and $\epsilon \lambda_{i+1}^T$ at the right, we find

$$X_{iA}{}^B = \langle ii + 1 \rangle \begin{pmatrix} -x_i^- \epsilon & -x_i^- \epsilon x_i^+ & -x_i^- \epsilon \theta_i \\ \epsilon & \epsilon x_i^+ & \epsilon \theta_i \\ -\varpi \theta_i^T \epsilon & -\varpi \theta_i^T \epsilon x_i^+ & -\varpi \theta_i^T \epsilon \theta_i \end{pmatrix}. \quad (3.31)$$

The $(4|6) \times (4|6)$ matrix X_i is such that $(X_i)^2 = 0$ and $\text{str } X_i = 0$.

From the definition it is clear that under superconformal transformations $h \in OSp(6|4)$ a matrix X transforms as $X \rightarrow h X h^{-1}$. Then, given two points X_i, X_j , we can construct an invariant $\text{str } X_i X_j$. Explicit computation yields

$$\text{str } X_i X_j = 2 \langle ii + 1 \rangle \langle jj + 1 \rangle \det \Delta x_{ij} = -2 \langle ii + 1 \rangle \langle jj + 1 \rangle (\Delta x_{ij})^2. \quad (3.32)$$

A three-point invariant is given by

$$\frac{\text{str}(X_i X_j X_k)}{\langle ii + 1 \rangle \langle jj + 1 \rangle \langle kk + 1 \rangle} = \text{tr}(\epsilon \Delta x_{ij} \epsilon \Delta x_{jk} \epsilon \Delta x_{ki}), \quad (3.33)$$

which is a supersymmetrization of $\epsilon_{\mu\nu\rho} x_{ij}^\mu x_{jk}^\nu x_{ki}^\rho \sim \epsilon_{\mu\nu\rho} x_i^\mu x_j^\nu x_k^\rho$.

So far have taken Z and \bar{Z} to have complex components. However, the $OSp(6|4)$ elements are represented by $(4|6) \times (4|6)$ supermatrices with real entries. We want the matrix $X_i / \langle ii + 1 \rangle$ to have real entries. This does not mean that the components of the twistors Z should be real. In fact, some of them have to be imaginary in order for the Wilson loop to close.

Later we will compute perturbative corrections to a polygonal light-like Wilson loop in superspace. This is a sum of diagrams with gluon and matter exchanges between the sides of the polygon and interaction vertices which are integrated over. There are several ways to parametrize the sides. We can take

$$X(\tau) = (1 - \tau) \frac{X_i}{\langle ii + 1 \rangle} + \tau \frac{X_{i+1}}{\langle i + 1i + 2 \rangle}, \quad (3.34)$$

⁷Writing the indices explicitly this becomes $X_{iA}{}^B = \bar{Z}_{i,A} Z_{i+1}^B - \bar{Z}_{i+1,A} Z_i^B$. Note that the fermionic components of Z_i and Z_{i+1} do not commute and the order is important.

where $\tau \in [0, 1]$. This parametrization is arranged such that $\text{str}(X(\tau)\mathcal{I}) = 1$. Then, the parts in the denominator of the propagator which involve the infinity twistor cancel out, but the rest of the terms become more complicated (and still depend on the infinity twistor). Another parametrization we can choose is⁸

$$X(t) = X_i + tX_{i+1}, \quad (3.35)$$

with $t \in [0, \infty)$.

Some of the diagrams contain integration over the space-time. Even though we will not consider in detail such diagrams in this paper, it is useful to work out the twistor representation of the integration measure. The superconformal integration measure in variables (x, θ) can be obtained as follows. To the point in superspace with coordinates (x, θ) we can associate two supertwistors (Z_A, Z_B) (here A and B are labels, not components of the a twistor Z). The over-lined versions of these supertwistors are not independent but can be obtained by transposition followed by multiplication by a constant matrix. These supertwistors satisfy the constraints $Z_A \cdot \bar{Z}_A = 0$, $Z_A \cdot \bar{Z}_B = 0$, $Z_B \cdot \bar{Z}_A = 0$ and $Z_B \cdot \bar{Z}_B = 0$. The only independent nontrivial constraint is $Z_A \cdot \bar{Z}_B = 0$. The choice of (Z_A, Z_B) is not unique; any other choice obtained by a $GL(2)$ transformation yields the same point in superspace. Moreover, the $GL(2)$ transformations preserve the constraint. Therefore, we need to divide by the action of this group. Using these ingredients we get

$$d^{3|12}X = \frac{d^{4|6}Z_A d^{4|6}Z_B}{\text{vol}(GL(2))} \delta(Z_A \cdot \bar{Z}_B). \quad (3.36)$$

More precisely, the measure $d^{4|6}Z_A d^{4|6}Z_B \delta(Z_A \cdot \bar{Z}_B)$ is invariant under $SL(2)$ but not under the $GL(2)$. In order to study the invariance under a global rescaling $X \rightarrow \lambda X$ we can rescale $Z_A \rightarrow \lambda Z_A$ and leave the Z_B invariant. Then, $d^{4|6}Z_A \rightarrow \lambda^{-2} d^{4|6}Z_A$ and a further contribution from δ produces a factor of λ^{-3} . Then we have that $d^{3|12}X \rightarrow \lambda^{-3} d^{3|12}X$. Without supersymmetry we would have a conformal invariant measure such that $d^3X \rightarrow \lambda^3 d^3X$. When gauging the extra degree of freedom in $GL(2)/SL(2)$ we consider that the measure is multiplied by a function such that this extra charge vanishes. We should further note that for $\mathcal{N} = 3$ supersymmetries the measure is exactly invariant under $GL(2)$ transformations.

This superconformal measure can be written in terms of (x, θ) variables by gauging $\lambda_A = (1, 0)$ and $\lambda_B = (0, 1)$. If we restrict to the bosonic case for simplicity and we set $Z_A = (1, 0, \mu_A)$ and $Z_B = (0, 1, \mu_B)$ then we can remove the group $GL(2)$ and we are left with

$$d^2\mu_A d^2\mu_B \delta(\mu_{A,2} - \mu_{B,1}). \quad (3.37)$$

⁸If we insist on using kinematics where the coordinates of the Wilson loop vertices are real, then some of the λ_i have to be imaginary. So in some cases τ goes from zero to infinity along the imaginary axis.

In this gauge the μ components are given by $\mu_A = (x_{11}, x_{12})$ and $\mu_B = (x_{21}, x_{22})$ and the constraint imposes the symmetry of the matrix x . Therefore,

$$d^3x = \frac{d^4z_A d^4z_B}{\text{vol}(GL(2))} \delta(z_A \cdot \bar{z}_B), \quad (3.38)$$

where we have denoted by lowercase letters the bosonic components of the super-twistors Z . We have also computed the normalization factor of the twistorial measure.

4 Super-Wilson loops

Let us now define the super-Wilson loops and show that they are classically invariant under superconformal transformations.

In order to define the polygonal super-Wilson loop we need a choice of contour and a connection. The superspace connection is given by $\mathcal{A} = e_B \cdot A_B + e_F \cdot A_F$, where e_B, e_F are the bosonic and the fermionic vielbeine and A_B, A_F are the bosonic and fermionic connection. This connection \mathcal{A} transforms nicely under super-gauge transformations.

The contour can be described by giving a sequence of light-like lines which intersect pairwise. As we described in sec. 3, each light-like line is parametrized by a pair of twistors (Z_i, \bar{Z}_i) such that $Z_i \cdot \bar{Z}_i = 0$. The conditions that the neighboring sides i and $i + 1$ of the polygon intersect is encoded in the constraint $Z_i \cdot \bar{Z}_{i+1} = 0$ which also implies $Z_{i+1} \cdot \bar{Z}_i = 0$.

However, as we also discussed in sec. 3, in space-time the right notion of light-like line has several fermionic directions (it is “fat” in the language of refs. [21, 51]). These fat lines intersect in points in superspace, which are the vertices of the polygonal contour.

Since the superspace connection \mathcal{A} is a 1-form, it needs to be integrated over a one-dimensional curve, but which one? The answer, which was given in ref. [21], is that, as long as the contour lies in the “fat” lines, it does not matter which contour we choose since the gauge field is flat there. In fact, for any choice of contour a conformal transformation will not preserve it (it only preserves the “fat” line) so in order to show invariance under conformal transformations we have to deform the contour to the previous one, while staying inside the “fat” line.

In the case of ABJM theory the field strength is flat on the “fat” lines, as one can see by examining the field strengths in eqs. (2.36) (2.37) (2.38).

Given the polygonal contour \mathcal{C} we can define the super-Wilson loop as

$$\mathcal{W} = \left\langle \text{tr} P \exp \left(\int_{\mathcal{C}} \mathcal{A} \right) \right\rangle. \quad (4.1)$$

Recall that the gauge connection \mathcal{A} lives in the $(\mathbf{Ad}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{Ad})$ of $U(N) \times U(M)$. Then, the trace above is taken in the $\mathbf{N} \oplus \mathbf{M}$ representation. However, this is not

the only option; one could instead consider the supertrace (see the discussions in refs. [52–58]).

We will restrict to the planar limit, where $N \rightarrow \infty$, $M \rightarrow \infty$, $k \rightarrow \infty$ such that the ratios $\frac{N}{k}$ and $\frac{M}{k}$ are constant. From our definition of the super-Wilson loop it follows that the $U(N)$ and the $U(M)$ parts do not mix. More explicitly, at the leading order in N and M all the Feynman graphs attach either to the $U(N)$ or to the $U(M)$ part of the super-Wilson loop. Since there are always two possibilities for the color factors and since they are very simple, we will not write them out explicitly.

Notice that the θ expansion of the connections in eqs. (2.34) (2.35) contains composite fields, i.e. products of local fields at the same space-time point. Such products are singular in the quantum theory and need to be normal-ordered. In fact, each light-like side needs to be normal-ordered since any contraction between fields on each side is singular. We will not attempt to give a prescription for how to do this.

The fact that at leading order in the color factors the $U(N)$ and $U(M)$ factors do not mix is not a satisfactory feature. One can introduce mixing in several ways. For example, we can insert bifundamental fields at the vertices. Another way, inspired by the construction (see ref. [55]) of the $\frac{1}{2}$ -BPS Wilson loop in ABJM theory, is to use a super-connection⁹ in the sense that the gauge part is a $(N|M) \times (N|M)$ supermatrix.

A natural question is whether the supersymmetric Wilson loop we construct in this paper is dual to scattering amplitudes. The answer is negative for several reasons. First, the symmetries do not match; the scattering amplitudes only preserve an $SU(3) \times U(1)$ out of the $SO(6)$ R-symmetry, while the supersymmetric Wilson loop we construct preserves the full $SO(6)$ symmetry. Second, the supersymmetric Wilson loop can be constructed for any number of sides while scattering amplitudes only exist for an even number of external particles. This can be cured by introducing operators transforming in bi-fundamental representations at the vertices of the Wilson loop. These operators can also naturally break the symmetry $SO(6) \rightarrow SU(3) \times U(1)$. The third reason why a duality with scattering amplitudes is not possible for the supersymmetric Wilson loops we study in this paper is the non-chiral nature of the $\mathcal{N} = 6$ superspace. The analog problem for $\mathcal{N} = 4$ super-Yang-Mills was studied in ref. [21, 51]. There it was shown that the superspace torsion prevents an identification between the kinematics of supersymmetric Wilson loops and scattering amplitudes, since it is not possible to define supersymmetric intervals that are both null and sum to zero. Chiral superspace is torsionless and there this obstruction disappears.

⁹This idea arose in conversations with Niklas Beisert.

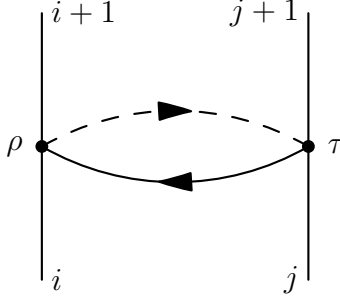


Figure 1: The lowest order contribution in the coupling to the $\chi_{i+1} \cdot \tilde{\chi}_{j+1}$ coefficient of the super-Wilson loop. The fermion propagators are solid lines while scalar propagators are dashed lines.

5 Some perturbative computations

What is the θ dependence of the supersymmetric Wilson loop? At the lowest order we need to compute a correlation between the terms inside $A_{\alpha\beta}$ which are linear in θ . We therefore need to compute correlation functions of type $\langle \psi(x) \bar{\phi}(x) \phi(y) \bar{\psi}(y) \rangle = \langle \psi(x) \bar{\psi}(y) \rangle \langle \bar{\phi}(x) \phi(y) \rangle$.

If we drop the color dependence the two-point functions for the scalars and fermions are

$$\langle \bar{\phi}_I(x) \phi^J(y) \rangle = -\frac{1}{4\pi} \delta_I^J \frac{1}{|x-y|}, \quad (5.1)$$

$$\langle \psi_J^\alpha(x) \bar{\psi}^{I\beta}(y) \rangle = -\frac{1}{8\pi} \delta_J^I \frac{(x-y)^{\alpha\beta}}{|x-y|^3}. \quad (5.2)$$

Due to the normalization $\text{tr } T^a T^b = -\frac{1}{2} \delta^{ab}$ of the gauge algebra generators, the color contribution can be obtained by multiplying by -2 for each propagator and by N for every loop transforming in the fundamental of $U(N)$ and by M for every loop transforming in the fundamental of $U(M)$.

We have several types of contributions: between the bilinear transforming in the adjoint of $U(N)$ and itself, subleading contributions between the bilinears transforming in the adjoint of $U(N)$ with $U(M)$, etc. Only the color structures are different between the leading color contributions. We will focus on the dependence on the odd variables.

When restricted to the light-like segment between points labeled by i and $i+1$ the terms in the superconnection which are linear in θ contribute (see fig. 1)

$$\begin{aligned} -\frac{i}{4g} d\tau \lambda_{i+1}^\beta \lambda_{i+1}^\gamma \left(\theta_\beta^{IJ}(\tau) \psi_{I\gamma}(\tau) \bar{\phi}_J(\tau) + \bar{\theta}_{IJ\beta}(\tau) \phi^I(\tau) \psi_\gamma^J(\tau) \right) = \\ = -\frac{id\tau}{4g} \left(\chi_{i+1}^{IJ} \lambda_{i+1}^\gamma \psi_{I\gamma}(\tau) \bar{\phi}_J(\tau) + \tilde{\chi}_{i+1, IJ} \phi^I(\tau) \bar{\psi}_\gamma(\tau) \lambda_{i+1}^\gamma \right), \end{aligned} \quad (5.3)$$

where $\theta_i(\tau) = (1 - \tau)\theta_i + \tau\theta_{i+1}$ and $\phi(\tau) = \phi((1 - \tau)x_i + \tau x_{i+1})$, etc.

If we compute the correlation between the parts linear in θ on the sides $(i, i + 1)$ and $(j, j + 1)$, where the side $(j, j + 1)$ is parametrized by σ , we find

$$- \frac{1}{(16\pi g)^2} \frac{(\chi_{i+1} \cdot \tilde{\chi}_{j+1})(z_{i+1} \cdot \tilde{z}_{j+1})}{(((1 - \tau)x_i + \tau x_{i+1} - (1 - \sigma)x_j - \sigma x_{j+1})^2)^2}, \quad (5.4)$$

where we have denoted by z_i the bosonic components of the supertwistor Z_i .

We further need to integrate this over τ and σ from 0 to 1. The integral is easy to do, and we finally get¹⁰

$$\frac{1}{(16\pi g)^2} \frac{(\chi_{i+1} \cdot \tilde{\chi}_{j+1})}{z_{i+1} \cdot \tilde{z}_{j+1}} \ln \left(\frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2} \right), \quad (5.5)$$

where we have used the identity $(z_{i+1} \cdot \tilde{z}_{j+1})^2 = x_{ij}^2 x_{i+1,j+1}^2 - x_{i,j+1}^2 x_{i+1,j}^2$.

This answer is a component of the expansion of the n -sided light-like supersymmetric Wilson loop $\mathcal{W}(Z_1, \dots, Z_n)$. Superconformal invariance dictates that \mathcal{W} should depend¹¹ only on terms like $Z_i \cdot \bar{Z}_j$ and terms of type $\chi_i \cdot \tilde{\chi}_j$ can only originate from $Z_i \cdot \bar{Z}_j$ terms. Therefore, the coefficient of $\chi_i \cdot \tilde{\chi}_j$ can be written as

$$\text{Coefficient of } \chi_i \cdot \tilde{\chi}_j \text{ in } \mathcal{W} = \left. \frac{\partial \mathcal{W}}{\partial Z_i \cdot \bar{Z}_j} \right|_{\chi=0}. \quad (5.6)$$

Using the result computed above we find that

$$d\mathcal{W}|_{\chi=0} \propto \frac{1}{(16\pi g)^2} \ln \left(\frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2} \right) d \ln(z_{i+1} \cdot \tilde{z}_{j+1}). \quad (5.7)$$

The proportionality factor is there because we have not included the color factors. This result can be thought as a differential equation which can be integrated in terms of dilogarithms.¹²

Up to coupling factors this result is exactly the same¹³ as for 1-loop Wilson loop in $\mathcal{N} = 4$ super-Yang-Mills, but here it appears at two loops. The first computation of this answer in ABJM theory was done in ref. [59]. Our derivation of the same result is simpler.

¹⁰This result holds if i and j are not neighbors, i.e. $i \neq j, j \pm 1$. If they are neighbors, then the integrand is exactly zero. This is unlike in the case of the $\mathcal{N} = 4$ super-Yang-Mills super-Wilson loop, where such contributions are UV-divergent.

¹¹The terms of type $z_i \cdot \tilde{z}_j$ are not all independent. They are related by Pfaffian constraints, which are the analog of Gram determinant constraints for an antisymmetric “metric”. As an example of constraint, consider six bosonic twistors z_i , $i = 1, \dots, 6$ and form the antisymmetric matrix $\mathcal{M} = (z_i \cdot \tilde{z}_j)_{i,j=1,\dots,6}$. Then, $\text{pf } \mathcal{M} = 0$ which imposes cubic constraints on the products $z_i \cdot \tilde{z}_j$.

¹²Of course, this method will not reproduce an additive constant. The constant can be obtained by using the properties of the Wilson loop under different degenerations.

¹³This form can be found for example in ref. [51, eqs. 6.12, 6.13]

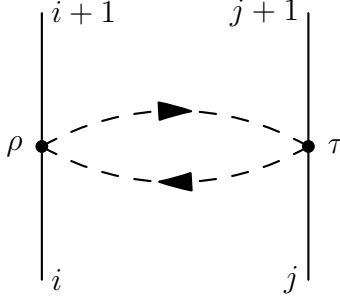


Figure 2: The two-scalar exchange contribution to the super-Wilson loop. It is of order four in the θ expansion.

We see that we have a mixing between contributions of different complexities but whose dependence on the coupling constant is the same. For example, two-loop diagrams with gauge fields and diagrams with matter fields which are of the same complexity as one loop, contribute at the same order in perturbation theory and are related by supersymmetry.

Let us now discuss the answer we obtain from the correlation functions involving scalar bilinears (see fig. 2). The relevant terms we are interested in appear in the odd part of the connection and read

$$-\frac{i}{g}e_F^{KL\beta}\bar{\theta}_{MK\beta}\left(\phi^M\bar{\phi}_L-\frac{1}{4}\delta_L^M\phi^P\bar{\phi}_P\right), \quad (5.8)$$

where e_F is the fermionic vielbein $e_F^{KL\beta} = d\theta^{KL\beta}$. If we restrict this to a light-like line between points (x_i, θ_i) and (x_{i+1}, θ_{i+1}) , with parametrization $\theta(\tau) = \theta_i + \tau(\theta_{i+1} - \theta_i)$, we obtain $e_F^{KL\beta} = -d\tau\lambda_{i+1}^\beta\eta_{i+1}^{LN}$. When dotted into $\bar{\theta}(\tau)$ the factor λ_{i+1} makes the dependence on τ disappear and we are left with

$$d\tau\bar{\theta}_{i,MK\beta}\lambda_{i+1}^\beta\eta_{i+1}^{KL} = d\tau\tilde{\chi}_{i+1,MK}\eta_{i+1}^{KL}. \quad (5.9)$$

Now we take two sides between vertices $(i, i+1)$ and $(j, j+1)$ and compute the correlation function between the bosonic bilinears. The position on the first line is parametrized by τ and on the second line by σ . If we label the position at which the fields are evaluated by τ and σ we find

$$d\tau\tilde{\chi}_{i+1,MK}\eta_{i+1}^{KN}d\sigma\tilde{\chi}_{j+1,PL}\eta_{i+1}^{LQ}\frac{\delta_Q^M\delta_N^P-\frac{1}{4}\delta_M^N\delta_P^Q}{(x(\tau)-x(\sigma))^2}. \quad (5.10)$$

The η can be expanded in terms of χ so everything can be written in terms of twistor components.

This formula produces an unusual contraction pattern of the $SU(4)$ indices $\tilde{\chi}_{i+1,MK}\eta_{i+1}^{KN}\tilde{\chi}_{j+1,NL}\eta_{i+1}^{LM}$. Superconformal invariants are built out of products $Z_i \cdot \bar{Z}_j$ whose nilpotent part is $\chi_{i,MN}\tilde{\chi}_j^{MN}$. At first sight it looks like the combination above

can not be written in this form. Nevertheless, we will show that it is possible to rewrite it in such a form. For this we consider two Grassmann variables ψ_{MN} , χ_{PQ} which are antisymmetric in the exchange $M \leftrightarrow N$ and $P \leftrightarrow Q$. Then, we have the following identity

$$\begin{aligned} \psi_{MN}\chi_{PQ} - \psi_{MP}\chi_{NQ} + \psi_{MQ}\chi_{NP} + \psi_{NP}\chi_{MQ} - \\ \psi_{NQ}\chi_{MP} + \psi_{PQ}\chi_{MN} = \frac{1}{4}\epsilon_{MNPQ}\epsilon^{RSTU}\psi_{RS}\chi_{TU}. \end{aligned} \quad (5.11)$$

This identity can be proved by noticing that the left-hand side is completely antisymmetric in $MNPQ$ and therefore it is proportional to ϵ_{MNPQ} . The proportionality constant can be obtained by contracting with ϵ^{MNPQ} .

Using this identity we can show that, given two other Grassmann variables A and B which are also antisymmetric in their two $SU(4)$ indices, we have

$$(\psi A \chi B) - (\psi B)(A \chi) - (\psi A)(\chi B) = -(\psi \chi)(AB), \quad (5.12)$$

where $\psi A \chi B = \psi_{IJ}A^{JK}\chi_{KL}B^{LI}$ and, as always, we normalize the index contractions such that $\psi A = \frac{1}{2}\psi_{IJ}A^{IJ}$. This formula allows us to rewrite the final answer in terms of objects whose origin is more clear from the point of view of superconformal symmetry.

6 Perturbative expansion vs. Grassmann expansion

We now want to understand how to compute the perturbative order of a given planar diagram contributing to the expectation value of a Wilson loop. In particular, we would like to relate the perturbative order with the topology of the diagram.

The most convenient way to explicit this relation is by rescaling the matter fields by a factor of \sqrt{g} . This way, all the interaction vertices contribute a factor of g and all propagators a factor of g^{-1} . Most importantly, with this rescaling the superconnection does not contribute any power of g , since this rescaling cancels the g^{-1} dependence in the Grassmann expansion. The perturbative order j associated with a given graph is then easily expressed in terms of the total number of propagators P and interaction vertices V_{int} as

$$j = V_{int} - P \quad (6.1)$$

We are interested in polygonal Wilson loops in the planar limit, thus homotopic to a disk. The Euler characteristic is 1, therefore the total number of vertices V , of edges E and of faces F satisfy

$$V - E + F = 1 \quad (6.2)$$

$g^\# \backslash \theta^\#$	0	1	2	3	4	5
0	0	a	0	0	0	0
-1	0	ϕ^2	$\phi\psi$	$\nabla\phi^2, \psi^2$	$\nabla\phi\psi$	$\nabla^2\phi^2, \nabla\psi^2$
-2	0	0	0	ϕ^4	$\phi^3\psi$	$\nabla\phi^4$
-3	0	0	0	0	0	$\phi^6, \phi^2\psi^2$

Table 1: The interplay between the θ expansion and the expansion in the coupling for the fermionic gauge connection.

$g^\# \backslash \theta^\#$	0	1	2	3	4	5
0	a	0	0	0	0	0
-1	0	$\phi\psi$	$\nabla\phi^2, \psi^2$	$\nabla\phi\psi$	$\nabla^2\phi^2, \nabla\psi^2$	$\nabla^2\phi\psi$
-2	0	0	ϕ^4	$\phi^3\psi$	$\phi^2\psi^2, \nabla\phi^4$	$\nabla\phi^3\psi, \phi\psi^3$
-3	0	0	0	0	$\phi^6, \phi^2\psi^2$	$\psi\phi^5$

Table 2: The interplay between the θ expansion and the expansion in the coupling for the bosonic gauge connection.

The total number of vertices is given by the number of interaction vertices plus the number of operator insertions on the boundary, $V = V_{int} + V_\partial$. Similarly, the number of edges is equal to the number of propagators plus the number of segments in which the boundary is partitioned by the insertions, $E = P + E_\partial$. Obviously, $V_\partial = E_\partial$, therefore the Euler characteristic equation becomes

$$V_{int} - P = 1 - F \quad (6.3)$$

In the right hand side one can recognize the perturbative order j . Therefore, given a graph with F faces, it contributes at order g^{1-F} .

The θ expansion of the gauge connections can be done explicitly to arbitrarily high order. Here we only want to comment on some of the qualitative features of this expansion. We have that a term of schematic form $\nabla^k \phi^l \psi^m \in A_F$ with $l + m$ even and $l + m \geq 2$ contributes to orders $\theta^{2k+l+2m-1}$ and $g^{-(l+m)/2}$. Terms of schematic type $\nabla^n \phi^p \psi^q \in A_B$ with $p + q$ even and $p + q \geq 2$ contribute to orders $\theta^{2n+p+2q-2}$ and $g^{-(p+q)/2}$. The types of fields resulting from this double expansion are listed in tables 1 2.

7 A peek at three loops

Let us now consider the three-loop, i.e. g^{-3} contributions to the super-Wilson loop. Among these contributions, we can look at terms with Grassmann weight zero, two,

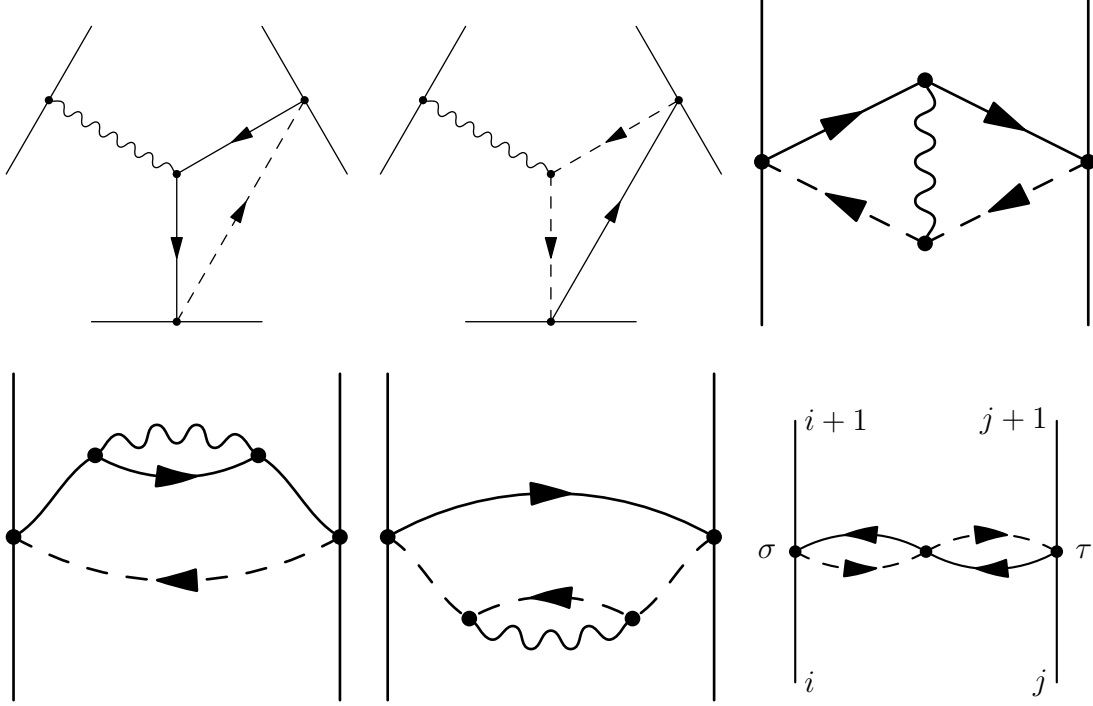


Figure 3: The diagrams contributing to order g^{-3} and to Grassmann weight two.

four, etc. We will see that the complexity of the computation decreases as we increase the Grassmann weight. For example, in fig. 3 we show the diagrams which contribute to the Grassmann weight two part and in fig 4 we show the diagram which contribute to the Grassmann weight four part. Needless to say, the Grassmann order zero part is a sum of many more diagrams.

For the contribution of the diagram in fig. 4 we obtain after a fermionic rearrangement as in eq. (5.12) and partial translation to twistor language

$$\frac{z_{j+1} \cdot \tilde{z}_{k+1}}{2(16\pi g)^3} \int_0^1 d\rho \int_0^1 d\tau \int_0^1 d\sigma \frac{(\chi_{i+1} \cdot \tilde{\chi}_{j+1})(\eta_{i+1} \cdot \tilde{\chi}_{k+1}) - (j \leftrightarrow k)}{|x_i(\rho) - x_j(\tau)| |x_i(\rho) - x_k(\sigma)| |x_j(\tau) - x_k(\sigma)|^3}, \quad (7.1)$$

where $x_i(\rho) = (1 - \rho)x_i + \rho x_{i+1}$, etc.

Using the expression for η_{i+1} we see that this result contains several types of quartic terms in χ . The terms of type $(\chi_{i+1} \cdot \tilde{\chi}_{j+1})(\chi_{i+1} \cdot \tilde{\chi}_{k+1})$ cancel by antisymmetry in variables j and k . Up to symmetries, the only other type of entry is $(\chi_i \cdot \tilde{\chi}_{j+1})(\chi_{i+1} \cdot \tilde{\chi}_{k+1})$. It appears in a combination

$$\frac{z_{j+1} \cdot \tilde{z}_{k+1}}{2(16\pi g)^3 \langle ii+1 \rangle} (\chi_i \cdot \tilde{\chi}_{j+1})(\chi_{i+1} \cdot \tilde{\chi}_{k+1}) \int_0^1 d\rho \int_0^1 d\tau \int_0^1 d\sigma \frac{1}{|x_j(\tau) - x_k(\sigma)|^3} \times \\ \left(\frac{1}{|x_{i-1}(\rho) - x_j(\sigma)| |x_{i-1}(\rho) - x_k(\tau)|} - \frac{1}{|x_i(\rho) - x_j(\sigma)| |x_i(\rho) - x_k(\tau)|} \right). \quad (7.2)$$

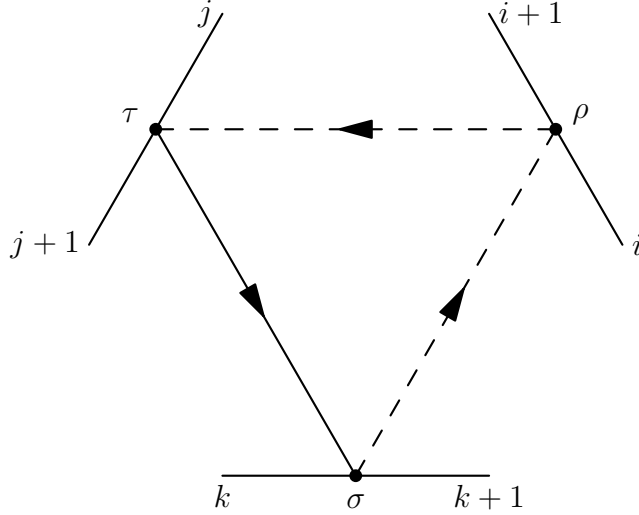


Figure 4: A diagram contributing to order g^{-3} and to Grassmann weight four. There are some other possibilities involving two scalar exchanges interacting with photons which can be obtained by replacing the fermion line in fig. 3 by a scalar line. However, if the i , j and k sides of the Wilson loop are separated by one or more sides, such diagrams do not contribute the same kind of Grassmannian quartic terms.

The integrals above are of the form

$$\int_0^1 d\rho \int_0^1 d\tau \int_0^1 d\sigma P_1(\tau, \rho)^{-1/2} P_2(\rho, \sigma)^{-1/2} P_3(\sigma, \tau)^{-3/2}, \quad (7.3)$$

where $P_i(x, y) = A_i xy + B_i x + C_i y + D_i$ and A_i, B_i, C_i, D_i are some constants. Two of these integrals can be performed using the formulas

$$\int_0^1 \frac{dt}{\sqrt{(mt+n)(pt+q)^3}} = -\frac{2}{np-mq} \left(\frac{\sqrt{m+n}}{\sqrt{p+q}} - \frac{\sqrt{n}}{\sqrt{q}} \right), \quad (7.4)$$

$$\int_0^1 \frac{dt}{\sqrt{at+b}(ct+d)\sqrt{et+f}} = \frac{1}{\sqrt{(ad-bc)(de-cf)}} \left[\ln\left(1 + \frac{c}{d}\right) - \right. \quad (7.5)$$

$$\left. 2 \ln\left(\frac{\sqrt{ad-bc}\sqrt{e+f} - \sqrt{de-cf}\sqrt{a+b}}{\sqrt{ad-bc}\sqrt{f} - \sqrt{de-cf}\sqrt{b}} \right) \right]. \quad (7.6)$$

At this stage, it appears unlikely that the third integral in eq. (7.3) can be computed in terms of classical polylogarithms.

In general, when integrating expressions of type $\int dt R(t) \ln S(t)$ where R and S are rational fractions in t the result can be expressed in terms of dilogarithms. However, here we have more complicated expressions due to the presence of square roots.

It is noteworthy that when performing the integrals as described above we obtain the square root of a cubic polynomial in the last integration step. Is this a hint that the result is an elliptic polylogarithm? For arbitrary values of A_i, B_i, C_i, D_i this cubic polynomial is generic, but for six points $x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}$ there is a Gram determinant constraint which imposes a constraint among the 12 values A_i, B_i, C_i, D_i . It is curious that when this constraint is satisfied the cubic polynomial mentioned above factorizes into a linear polynomial and the square of another linear polynomial.

Another way to understand this constraint is to think more deeply about the geometry of the problem. We have three lines containing points (x_i, x_{i+1}) , (x_j, x_{j+1}) and (x_k, x_{k+1}) respectively. Let us now find the transversals, i.e. the light-like lines intersecting all of three lines (an analogous problem in 4D was considered in refs. [60, 61]). To find a light-like line intersecting lines (x_i, x_{i+1}) , (x_j, x_{j+1}) and (x_k, x_{k+1}) we pick a point on each one of them with parameter ρ , σ and τ respectively. Then, the light-like conditions read $P_1(\tau, \rho) = 0$, $P_2(\rho, \sigma) = 0$ and $P_3(\sigma, \tau) = 0$. The first two equations are linear in τ and σ so we can trivially solve for them and plug back in the third. We obtain a degree two polynomial whose discriminant is zero since it is equal to the Gram determinant constraint for six points in three dimensions.

Therefore, in three dimensions, there is a unique (with multiplicity two) light-like line intersecting any three non-intersecting light-like lines. If we denote by ρ_* , σ_* and τ_* the parameters of the intersection points and perform a change of variables $r = \rho - \rho_*$, $s = \sigma - \sigma_*$ and $t = \tau - \tau_*$ and we also rescale the polynomials P_i such as to make the leading coefficient equal to unity, then we reduce the problem to performing the integral

$$\int_{-\rho_*}^{1-\rho_*} dr \int_{-\sigma_*}^{1-\sigma_*} ds \int_{-\tau_*}^{1-\tau_*} dt Q_1(t, r)^{-1/2} Q_2(r, s)^{-1/2} Q_3(s, t)^{-3/2}, \quad (7.7)$$

where $Q_i(x, y) = xy + b_i x + c_i y$ and such that $b_1 b_2 b_3 + c_1 c_2 c_3 = 0$.

The integral in eq. (7.3) can be transformed by using the star-triangle identities (see ref. [62]). The triangle in eq. (7.3) is a semi-unique triangle in the language of this reference. It can be computed in terms of triangles with exponents $\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}$ and of stars with exponents 2, 1, 1. The triangles with those exponents can be computed in terms of dilogarithms. In the star integral all of the fractional powers disappear, and the integrals over ρ, τ and σ factorize and can be computed straightforwardly; however, the remaining integrals over the vertex of the star involve the square of a logarithm times a rational function. These integrals seem to be just as complicated as the original ones.

It is also worth mentioning that the integral in eq. (7.3) simplifies at low number of points, even though it remains still quite complicated. At low number of points, it is possible to translate some of the vertices to $\theta = 0$; due to this fact, supersymmetry is more powerful at higher number of points, therefore we focus on these cases.

We can try to compute the integral in fig. 4 in special kinematics. For example, if the sides $(i, i+1)$, $(j, j+1)$ and $(k, k+1)$ of the Wilson loop belong to the same two-dimensional plane the kinematics simplifies enough to allow an explicit evaluation of the integral and the answer is rational. This is not in contradiction with the fact that the g^{-3} part of the answer is of transcendentality three. Indeed, consider the dilogarithm $\text{Li}_2(x)$ whose derivative is $-\frac{\ln(1-x)}{x}$, whose limit is equal to 1 when evaluated at $x = 0$. Another example which is closer to the one we are considering here is to take two derivatives and one limit. If we do this for $\text{Li}_3(x)$ and we compute the limit $x \rightarrow 0$ we again obtain a rational answer.

Acknowledgments

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